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More on fuzzy lattices☆

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ABSTRACT

In this paper, the concept of an $(\in, \in \vee q_k)$-fuzzy sublattice (ideal, filter) and $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy sublattice (ideal, filter) in a lattice is introduced, which is a generalization of the concept as given by Davvaz and Kazancı(2011) [18]. Characterizations for an $(\in, \in \vee q_k)$-fuzzy sublattice (ideal, filter) and $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy sublattice (ideal, filter) are provided. Different classes of lattices are characterized by the properties of these fuzzy sublattices (ideals, filters). Using the notions of a fuzzy sublattice (ideal, filter) with thresholds, characterization of a fuzzy sublattice (ideal, filter), an $(\in, \in \vee q_k)$-fuzzy sublattice (ideal, filter) and $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy sublattice (ideal, filter) are discussed.

1. Introduction

Algebraic structures play a prominent role in mathematics with wide ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces and the like. This provides sufficient motivation to researchers to review various concepts and results from the realm of abstract algebra in the broader framework of fuzzy setting. One of the structures that are most extensively used and discussed in mathematics and its applications are lattices. As it is well known, it is considered as a relational, ordered structure, on one hand, and as an algebra, on the other hand (see e.g. [1,2]).

The theory of fuzzy sets which was introduced by Zadeh [3,4] is applied to many mathematical branches. Goguen generalized them to the notion of L-fuzzy sets [5]. On the other hand, few years after the inception of the notion of fuzzy set, Rosenfeld started the pioneer work in the domain of fuzzification of the algebraic objects, with his work on fuzzy groups [6]. This work is a contribution to the theory founded on the ideas of those authors and their followers. Das [7] characterized fuzzy subgroups by their level subgroups. In [8], Liu applied the concept of fuzzy sets to the theory of rings and introduced and examined the notion of a fuzzy ideal of a ring. A new type of fuzzy subgroup (viz, $(\in, \in \vee q)$-fuzzy subgroup) was introduced in an earlier paper of Bhakat and Das [9] by using the combined notions of "belongingness" and "quasicoincidence" of fuzzy points and fuzzy sets. In fact, $(\in, \in \vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. This concept has been studied further in [10–19]. Also, a generalization of Rosenfeld's fuzzy subgroup, and Bhakat and Das's fuzzy subgroup is given in [20]. In [21], Yuan Bo and Wu Wangming introduced the concept of fuzzy ideal of a distributive lattice. The definition of L-fuzzy lattices (fuzzy lattices valued by lattices) was introduced by Tepavcevic and Trajkovski in [22]. Also, see [23–27].

In this paper, we introduce the notion of $(\in, \in \vee q_k)$-fuzzy sublattice (ideal, filter) and $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy sublattice (ideal, filter) which is a generalization of an $(\in, \in \vee q)$-fuzzy sublattice (ideal, filter) in a lattice. We investigated related properties

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and provided characterizations of an $(\in, \in \vee q_k)$-fuzzy sublattice (ideal, filter) and $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$-fuzzy sublattice (ideal, filter). We discussed the implication-based fuzzy sublattice (ideal, filter) of a lattice. The important achievement of the study with an $(\in, \in \vee q_k)$-fuzzy sublattice (ideal, filter) is that the notion of an $(\in, \in \vee q)$-fuzzy sublattice (ideal, filter) is a special case of an $(\in, \in \vee q_k)$-fuzzy sublattice (ideal, filter), and thus many results in [18] are corollaries of our results obtained in this paper.

2. Preliminaries

In this section, we recall some definitions and results which will be used in what follows. Throughout the paper, we shall denote a lattice $(L, \vee, \wedge, 0, 1)$ by L, where the join and meet operations are denoted by $\vee$ and $\wedge$ in L, respectively. The top element of the lattice L is denoted by 1; similarly for the bottom element, 0.

Definition 2.1. Let L be a lattice and $L' \neq \emptyset$ be a subset of L such that for every pair of elements a, b in L' both $a \vee b$ and $a \wedge b$ are in L', then we say that L' with the same operations is a *sublattice* of L.

Definition 2.2. An *ideal* I of L is a non-empty subset of L such that

(i) $a, b \in I \Rightarrow a \vee b \in I$,
(ii) $a \in I$, $b \in L$ and $b \leq a \Rightarrow b \in I$.

A *filter* of L is the dual concept of an ideal.

Definition 2.3. A proper ideal I of L is called a *prime ideal* of L if $a, b \in L$ and $a \wedge b \in I$ imply $a \in I$ or $b \in I$. A *prime filter* of L is the dual concept of a prime ideal.

Throughout this paper $\langle [0, 1], \sqcup, \sqcap \rangle$ is a complete lattice, where $[0, 1]$ is the set of reals numbers between 0 and 1 and $x \sqcup y = \max\{x, y\}, x \sqcap y = \min\{x, y\}$.

Let X be a non-empty set. A fuzzy subset μ of X is a function $\mu : X \longrightarrow [0, 1]$. Let μ be any fuzzy subset of X. The set $\mu_t = \{x \in X \mid t \leq \mu(x)\}, t \in [0, 1]$, is called a *level subset* of μ.

In [21], Yuan Bo and Wu Wangming introduced the concept of fuzzy ideal of a distributive lattice. Also, see [23–27].

Definition 2.4. A fuzzy subset μ of L is called a *fuzzy sublattice* of L, if $\mu(x) \sqcap \mu(y) \leq \mu(x \vee y) \sqcap \mu(x \wedge y)$ for all $x, y \in L$.

Definition 2.5. (i) A fuzzy sublattice μ is called a *fuzzy ideal*, if $\mu(x \vee y) = \mu(x) \sqcap \mu(y)$ for all $x, y \in L$.
(ii) A fuzzy sublattice μ is called a *fuzzy filter*, if $\mu(x \wedge y) = \mu(x) \sqcap \mu(y)$ for all $x, y \in L$.

It is easy to see that a fuzzy sublattice μ of L is a fuzzy ideal (filter) of L if and only if $x \leq y$ implies that $\mu(x) \geq \mu(y)(\mu(x) \leq \mu(y))$ for all $x, y \in L$.

Definition 2.6. (i) A proper fuzzy ideal μ of L is called *fuzzy prime ideal*, if $\mu(x \wedge y) \leq \mu(x) \sqcup \mu(y)$ for all $x, y \in L$.
(ii) A proper fuzzy filter μ of L is called a *fuzzy prime filter*, if $\mu(x \vee y) \leq \mu(x) \sqcup \mu(y)$ for all $x, y \in L$.

We can easily obtain the following statement. A fuzzy subset μ is a fuzzy prime ideal if $\overline{\mu}$ is a fuzzy prime filter of L, where $\overline{\mu}$ is defined by setting $\overline{\mu}(x) = 1 - \mu(x)$ for all $x \in L$.

3. $(\in, \in \vee q_k)((\overline{\in}, \overline{\in} \vee \overline{q_k}))$-fuzzy sublattice (ideal, filter) of a lattice

For any fuzzy subset μ of L, the set $\{x \in L | 0 < \mu(x)\}$ is called the *support* of μ, and is denoted by *supp* μ. A fuzzy set μ, on L which takes the value $t \in (0, 1]$ at some $x \in L$ and takes the value 0 for all $y \in L$ expect x is called a *fuzzy point* and is denoted by x_t, where the point x is called its *support point* and t is called its *value*.

In what follows, let k denote an arbitrary element of $[0, 1)$ unless otherwise specified. For a fuzzy point, x_t has the following properties.

(1) It *belongs* to a fuzzy set μ, written as $x_t \in \mu$ if $\mu(x) \geq t$.
(2) It is *k-quasi-coincident* with a fuzzy set μ, written as $x_t q_k \mu$ if $\mu(x) + t + k > 1$.
(3) $x_t \in \mu$ or $x_t q_k \mu$, then we write $x_t \in \vee q_k \mu$.
(4) The formula $x_t \overline{\alpha} \mu$ means that $x_t \alpha \mu$ does not hold for $\alpha \in \{\in, q_k, \in \vee q_k\}$.

Definition 3.1. A fuzzy subset μ of L is called an $(\in, \in \vee q_k)$-*fuzzy sublattice* of L if for all $t, r \in (0, 1]$ and $x, y \in L$,

(i) $x_t, y_r \in \mu$ implies $(x \vee y)_{t \sqcap r} \in \vee q_k \mu$,
(ii) $x_t, y_r \in \mu$ implies $(x \wedge y)_{t \sqcap r} \in \vee q_k \mu$.

μ is called an $(\in, \in \vee q_k)$-*fuzzy ideal of* L if μ is an $(\in, \in \vee q_k)$-fuzzy sublattice of L and

(iii) $x_t \in \mu$ with $y \leq x$ implies $y_t \in \vee q_k \mu$,

μ is called an $(\in, \in \vee q_k)$-*fuzzy filter* of L, if μ is an $(\in, \in \vee q_k)$-fuzzy sublattice of L and

(iv) $y_t \in \mu$ and $y \le x$ implies $x_t \in \vee q_k \mu$.

An $(\in, \in \vee q_k)$-fuzzy sublattice (ideal, filter) of L with $k = 0$ is called an $(\in, \in \vee q)$-fuzzy sublattice (ideal, filter) of L (see [18, Definition 2.1]).

Theorem 3.2. *Conditions* (i)–(iv) *in Definition* 3.1 *are equivalent to the following conditions respectively.*

(1) $\mu(x) \sqcap \mu(y) \sqcap \frac{1-k}{2} \le \mu(x \vee y)$,
(2) $\mu(x) \sqcap \mu(y) \sqcap \frac{1-k}{2} \le \mu(x \wedge y)$,
(3) $y \le x$ *implies* $\mu(x) \sqcap \frac{1-k}{2} \le \mu(y)$,
(4) $y \le x$ *implies* $\mu(y) \sqcap \frac{1-k}{2} \le \mu(x)$,

for all $x, y \in L$.

Proof. (i) $\Longrightarrow$ (1): Suppose that $x, y \in L$. We consider the following cases:
(a) $\mu(x) \sqcap \mu(y) < \frac{1-k}{2}$,
(b) $\frac{1-k}{2} \le \mu(x) \sqcap \mu(y)$.

Case a: Assume that $\mu(x \vee y) < \mu(x) \sqcap \mu(y) \sqcap \frac{1-k}{2}$, which implies $\mu(x \vee y) < \mu(x) \sqcap \mu(y)$. Choose t such that $\mu(x \vee y) < t < \mu(x) \sqcap \mu(y)$. Then $x_t, y_t \in \mu$, but $(x \vee y)_t \overline{\in \vee q_k \mu}$ which contradicts (i).

Case b: Assume that $\mu(x \vee y) < \frac{1-k}{2}$. Then $x_{\frac{1-k}{2}}, y_{\frac{1-k}{2}} \in \mu$, but $(x \vee y)_{\frac{1-k}{2}} \overline{\in \vee q_k} \mu$, a contradiction. Hence (1) holds.

(ii) $\Longrightarrow$ (2): the proof is similar to (i) $\Longrightarrow$ (1).

(iii) $\Longrightarrow$ (3): let $x, y \in L$ and $y \le x$. We consider the following cases:
(a) $\mu(x) < \frac{1-k}{2}$,
(b) $\frac{1-k}{2} \le \mu(x)$.

Case a: Assume that $\mu(x) = t < \frac{1-k}{2}$ and $\mu(y) = r < \mu(x)$. Choose s such that $r < s < t$ and $r + s < 1 - k$. Then $x_s \in \mu$, but $y_s \overline{\in \vee q_k \mu}$ which contradicts (iii). So $\mu(x) \sqcap \frac{1-k}{2} \le \mu(y)$.

Case b: Let $\frac{1-k}{2} < \mu(x)$. If $\mu(y) < \mu(x) \sqcap \frac{1-k}{2}$, then $x_{\frac{1-k}{2}} \in \mu$, but $y_{\frac{1-k}{2}} \overline{\in \vee q_k \mu}$, which contradicts (iii). So $\mu(x) \sqcap \frac{1-k}{2} \le \mu(y)$.

(iv) $\Longrightarrow$ (4): the proof is similar to (iii) $\Longrightarrow$ (3).

(1 $\Longrightarrow$ (i)): let $x_t, y_r \in \mu$, then $t \le \mu(x)$ and $r \le \mu(y)$. Now, we have $t \sqcap r \sqcap \frac{1-k}{2} \le \mu(x) \sqcap \mu(y) \sqcap \frac{1-k}{2} \le \mu(x \vee y)$.

If $\frac{1-k}{2} < t \sqcap r$, then $\frac{1-k}{2} \le \mu(x \vee y)$ which implies that $1 - k < \mu(x \vee y) + t \sqcap r$.

If $t \sqcap r \le \frac{1-k}{2}$, then $t \sqcap r \le \mu(x \vee y)$. Therefore $(x \vee y)_{t \sqcap r} \in \vee q_k \mu$.

(2 $\Longrightarrow$ (ii)): the proof is similar to (1 $\Longrightarrow$ i).

(3 $\Longrightarrow$ (iii)): let $x_t \in \mu$. Then $t \le \mu(x)$. Now, we have $t \sqcap \frac{1-k}{2} \le \mu(x) \sqcap \frac{1-k}{2} \le \mu(y)$, which implies $t \le \mu(y)$ or $\frac{1-k}{2} \le \mu(y)$ according as $t \le \frac{1-k}{2}$ or $\frac{1-k}{2} < t$. Therefore $y_t \in \vee q_k \mu$.

(4 $\Longrightarrow$ (iv)): the proof is similar to (3 $\Longrightarrow$ (iii)). □

The above theorem is a generalization of Theorem 4.2 in [18].

The following corollary is exactly obtained from Definition 3.1 and Theorem 3.2.

Corollary 3.3. *Let* μ *be a fuzzy subset of a lattice* L. *Then*

(1) μ *is an* $(\in, \in \vee q_k)$-*fuzzy sublattice of* L *if and only if the conditions* (1) *and* (2) *in Theorem* 3.2 *hold;*
(2) μ *is an* $(\in, \in \vee q_k)$-*fuzzy ideal of* L *if and only if* μ *is an* $(\in, \in \vee q_k)$-*fuzzy sublattice, and the condition* (3) *in Theorem* 3.2 *hold;*
(3) μ *is an* $(\in, \in \vee q_k)$-*fuzzy filter of* L *if and only if* μ *is an* $(\in, \in \vee q_k)$-*fuzzy sublattice, and the condition* (4) *in Theorem* 3.2 *hold.*

Remark 3.4. Every fuzzy sublattice (ideal, filter) and $(\in, \in \vee q)$-fuzzy sublattice (ideal, filter) μ of L is an $(\in, \in \vee q_k)$-fuzzy sublattice (ideal, filter) of L, but, as the following example shows, the converse is not necessarily true.

Example 3.5. Consider the lattice $L = \{0, a, b, 1\}$ whose Hasse diagram is as given in Fig. 1.

Let $\mu: L \to [0, 1]$ be defined by $\mu(0) = 1$, $\mu(a) = 0.6$, $\mu(b) = 0.8$, $\mu(1) = 0.4$ Then

- μ is an $(\in, \in \vee q_{0.2})$-fuzzy ideal,
- μ is not an $(\in, \in \vee q)$-fuzzy ideal,
- μ is not a fuzzy ideal of L.

Theorem 3.6. *A non-empty subset* I *of* L *is a sublattice (ideal, filter) of* L *if and only if* χ_I *is an* $(\in, \in \vee q_k)$-*fuzzy sublattice (ideal, filter) of* L.

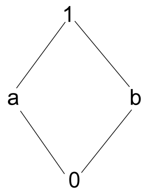

Fig. 1.

Proof. Assume that I is a sublattice (ideal, filter) of L. Then χ_I is a fuzzy sublattice (ideal, filter) in the sense of Definition 3.1 and so it is an $(\in, \in \vee q_k)$-fuzzy sublattice (ideal, filter).

For the converse, only we prove the case that χ_I is an $(\in, \in \vee q_k)$-fuzzy ideal of L. In this case, for every $x, y \in I$, we have $\frac{1-k}{2} = \chi_I(x) \sqcap \chi_I(y) \sqcap \frac{1-k}{2} \leq \chi_I(x \vee y)$ and so $x \vee y \in I$. Now, let $x \in I$, $y \in L$ and $y \leq x$. Then $\frac{1-k}{2} = \chi_I(x) \sqcap \frac{1-k}{2} \leq \chi_I(y)$ and so $y \in I$. □

As in Theorem 3.2, the above theorem is a generalization of Theorem 4.5 in [18].

Definition 3.7. A fuzzy subset μ of L is called an $(\in, \in \vee q_k)$-*fuzzy prime ideal of* L if μ is an $(\in, \in \vee q_k)$-fuzzy ideal of L and for all $t \in (0, 1]$ and $x, y \in L$, $(x \wedge y)_t \in \mu$ implies $x_t \in \vee q_k \mu$ or $y_t \in \vee q_k \mu$. An $(\in, \in \vee q_k)$-*fuzzy prime filter* of L is the dual concept of an $(\in, \in \vee q_k)$- fuzzy prime ideal.

An $(\in, \in \vee q_k)$-fuzzy prime ideal (filter) of L with $k = 0$ is called an $(\in, \in \vee q)$-fuzzy prime ideal (filter) of L (see [18, Definition 4.6]).

Theorem 3.8. *Let I be a prime ideal (filter) of a lattice L. Then χ_I is an $(\in, \in \vee q_k)$-fuzzy prime ideal (filter) of L.*

Proof. Let I be a prime ideal of L. Then χ_I is an $(\in, \in \vee q_k)$-fuzzy ideal of L. Now, let $(x \vee y)_t \in \chi_I$ then $\chi_I(x \vee y) = 1$ and so $x \vee y \in I$. Since I is a prime ideal, then $x \in I$ or $y \in I$. So $x_t \in \vee q_k \mu$ or $y_t \in \vee q_k \mu$. Hence χ_I is an $(\in, \in \vee q_k)$-fuzzy prime ideal. The proof for the case that I is a prime filter is similar. □

It is also easily seen that for $k = 0$, Theorem 4.7 in [18] is obtained.

Theorem 3.9. *Let I be a non-empty subset of a lattice L. If χ_I is an $(\in, \in \vee q_k)$-fuzzy prime ideal (filter) of L, then I is a prime ideal (filter) of L.*

Proof. According to Theorem 3.8, I is an ideal of L. We show that I is prime. Let $x, y \in L$ and $x \vee y \in I$. Then $\chi_I(x \vee y) = 1$. Since χ_I is an $(\in, \in \vee q_k)$-fuzzy prime ideal and $(x \vee y)_t \in \chi_I$ for all $t \in (0, 1]$, then $x_t \in \vee q_k \chi_I$ or $y_t \in \vee q_k \chi_I$. If $x_t \in \vee q_k \chi_I$, then $1 < \chi_I(x) + t + k$ and so $x \in I$. If $y_t \in \vee q \chi_I$, then $1 < \chi_I(y) + t + k$ and so $y \in I$. The proof for the case that $\chi_I(\in, \in \vee q_k)$-fuzzy prime filter is similar. □

If we take $k = 0$ in Theorem 3.9, then we have Theorem 4.8 in [18].

The following theorem is a more updated result than [18, Theorem 4.9].

Theorem 3.10. *A fuzzy subset μ of a lattice L is an $(\in, \in \vee q_k)$-fuzzy sublattice (ideal, filter) of L if and only if the set $\mu_t (\neq \emptyset)$ is a sublattice (ideal, filter) of L for all $0 < t \leq \frac{1-k}{2}$.*

Proof. Let μ be an $(\in, \in \vee q_k)$-fuzzy sublattice of L and $0 < t \leq \frac{1-k}{2}$. Let $x, y \in \mu_t$, then $t \leq \mu(x)$ and $t \leq \mu(y)$. Now, we have

$$t \leq t \sqcap \frac{1-k}{2} = \mu(x) \sqcap \mu(y) \sqcap \frac{1-k}{2} \leq \mu(x \vee y),$$
$$t \leq t \sqcap \frac{1-k}{2} = \mu(x) \sqcap \mu(y) \sqcap \frac{1-k}{2} \leq \mu(x \wedge y),$$

and so $x \vee y \in \mu_t$ and $x \wedge y \in \mu_t$. Hence μ_t is a sublattice of L.

Now, let μ be an $(\in, \in \vee q_k)$-fuzzy ideal of L. We suppose that $x \in \mu_t$, $y \in L$ and $y \leq x$. Then $t = t \sqcap \frac{1-k}{2} \leq \mu(x) \sqcap \frac{1-k}{2} \leq \mu(y)$ which implies $y \in \mu_t$. Hence μ_t is an ideal of L.

Conversely, let μ be a fuzzy subset of L such that μ_t $(\neq \emptyset)$ be a sublattice of L for all $0 < t \leq \frac{1-k}{2}$. For every $x, y \in L$, we can write

$$t_0 = \mu(x) \sqcap \mu(y) \sqcap \frac{1-k}{2} \leq \mu(x), \qquad t_0 = \mu(x) \sqcap \mu(y) \sqcap \frac{1-k}{2} \leq \mu(y),$$

then $x \in \mu_{t_0}$ and $y \in \mu_{t_0}$, so $x \vee y \in \mu_{t_0}$ and $x \wedge y \in \mu_{t_0}$. Therefore

$$\mu(x) \sqcap \mu(y) \sqcap \frac{1-k}{2} \leq \mu(x \vee y), \qquad \mu(x) \sqcap \mu(y) \sqcap \frac{1-k}{2} \leq \mu(x \wedge y)$$

which implies that μ is an $(\in, \in \vee q_k)$-fuzzy sublattice of L. Now, let μ_t be an ideal of L. We suppose that $x, y \in L$ and $y \leq x$. We have $t_0 = \mu(x) \sqcap \frac{1-k}{2} \leq \mu(x)$, so $x \in \mu_{t_0}$. Since μ_{t_0} is an ideal of L and $y \leq x$ we obtain that $y \in \mu_{t_0}$. Thus $t_0 \leq \mu(y)$ or $\mu(x) \sqcap \frac{1-k}{2} \leq \mu(y)$. Therefore μ is an $(\in, \in \vee q_k)$-fuzzy ideal of L.

The proof of the filter case is similar. □

Definition 3.11. A fuzzy subset μ of a lattice L is called an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-*fuzzy sublattice* of L if $(x \vee y)_{t \sqcap r} \overline{\in} \mu \Rightarrow x_t \overline{\in} \vee \overline{q_k}\mu$ or $y_r \overline{\in} \vee \overline{q_k}\mu$ and $(x \wedge y)_{t \sqcap r} \overline{\in} \mu \Rightarrow x_t \overline{\in} \vee \overline{q_k}\mu$ or $y_r \overline{\in} \vee \overline{q_k}\mu$ for all $x, y \in L$.

A fuzzy sublattice μ of a lattice L is called an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-*fuzzy ideal* of L if $y_t \overline{\in} \mu$ and $y \leq x$ implies $x_t \overline{\in} \vee \overline{q_k}\mu$ for all $x, y \in L$.

A fuzzy sublattice μ of a lattice L is called an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-*fuzzy filter* of L if $x_t \overline{\in} \mu$ and $y \leq x$ implies $y_t \overline{\in} \vee \overline{q_k}\mu$ for all $x, y \in L$.

An $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy sublattice (ideal, filter) of L with $k = 0$ is called an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy sublattice (ideal, filter) of L.

Every fuzzy sublattice (ideal, filter) is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy sublattice (ideal, filter) of L, but the converse may not be true as seen in the following example.

Example 3.12. Consider the lattice L which is given in Example 3.5. Let μ be a fuzzy subset of L defined by:

$$\mu(0) = 0.2, \qquad \mu(a) = 0.4, \qquad \mu(b) = 0.3, \qquad \mu(1) = 0.2.$$

It is easy to check that μ is an $(\overline{\in}, \overline{\in} \vee \overline{q_{0.2}})$-fuzzy ideal of L, but it is not a fuzzy ideal of L since $\mu(a \vee b) = 0.2 \neq 0.3 = \mu(a) \sqcap \mu(b)$.

Theorem 3.13. *A fuzzy subset μ of a lattice L is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy sublattice (ideal, filter) of L if and only if the set $\mu_t(\neq \emptyset)$ is a sublattice (ideal, filter) of L for all $\frac{1-k}{2} < t \leq 1$.*

Proof. It is similar to the proof of Theorem 3.10. □

Corollary 3.14. *For a fuzzy subset μ of a lattice L, the following are equivalent.*

(1) *μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy sublattice (ideal, filter) of L.*
(2) *$\mu_t(\neq \emptyset)$ is a sublattice (ideal, filter) of L for all $t \in (0.5, 1]$.*

Proof. The proof follows from Theorem 3.13, if we take $k = 0$. □

For a fuzzy subset μ of L, we consider the following set:

$K_t = \{t \in (0, 1] \mid \mu_t(\neq \emptyset) \Longrightarrow \mu_t$ is a fuzzy sublattice (ideal, filter) of $L\}$.

Then

- if $K_t = (0, 1]$, then μ is a fuzzy sublattice (ideal, filter) of L,
- if $K_t = (0, \frac{1-k}{2}]$, then μ is an $(\in, \in \vee q_k)$-fuzzy sublattice (ideal, filter) of L,
- if $K_t = (\frac{1-k}{2}, 1]$, then μ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy sublattice (ideal, filter) of L.

An obvious question is whether μ is a kind of fuzzy sublattice (ideal, filter) or not when $K_t \neq \emptyset$ (e.g., $K_t = (\alpha, \beta], \alpha, \beta \in (0, 1)\alpha < \beta$?).

Definition 3.15 (*[15, Definition 4.11]*)**.** Let $\alpha, \beta \in [0, 1]$ and $\alpha < \beta$. Let μ be a fuzzy subset of a lattice L. Then μ is called a *fuzzy sublattice with thresholds of* L, if for all $x, y \in L$,

(i) $\mu(x) \sqcap \mu(y) \sqcap \beta \leq \mu(x \vee y) \sqcup \alpha$,
(ii) $\mu(x) \sqcap \mu(y) \sqcap \beta \leq \mu(x \wedge y) \sqcup \alpha$.

Moreover, μ is a *fuzzy ideal with thresholds of* L, if and only if μ satisfies the conditions (i) and (ii) and satisfies the following condition:

(iii) $y \leq x$ implies $\mu(x) \sqcap \beta \leq \mu(y) \sqcup \alpha$.

Finally, μ is a *fuzzy filter with thresholds of* L, if and only if μ satisfies the conditions (i) and (ii) and satisfies the following condition:

(iv) $y \leq x$ implies $\mu(y) \sqcap \beta \leq \mu(x) \sqcup \alpha$.

The following example shows that there exist $\alpha, \beta \in (0, 1]$ with $\alpha < \beta$ such that μ is a sublattice (ideal, filter) with thresholds α, β which is not an $(\in, \in \vee q_k)$-fuzzy sublattice (ideal, filter) of L.

Example 3.16. Consider the lattice $L = \{0, a, b, c, d, e, f, 1\}$ whose Hasse diagram is as given in Fig. 2.

Let μ be a fuzzy subset of L defined by: $\mu(0) = 0$, $\mu(a) = \frac{2}{5}$, $\mu(b) = \frac{1}{6}$, $\mu(c) = \mu(d) = \frac{3}{5}$, $\mu(0) = \mu(e) = \mu(f) = \frac{5}{6}$.

Then it is easy to check that μ is a fuzzy ideal with thresholds $\alpha = 0.2$, $\beta = 0.6$. But it is not an $(\in, \in \vee q_{0.2})$-fuzzy ideal of L.

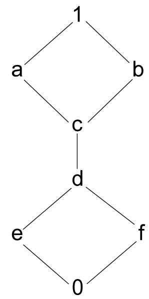

Fig. 2.

Now, we characterize fuzzy sublattice (ideal, filter) with thresholds by their level sublattice (ideal, filter).

Theorem 3.17. *A fuzzy subset μ of a lattice L is a fuzzy sublattice (ideal, filter) with thresholds of L if and only if μ_t ($\neq \emptyset$) is a sublattice (ideal, filter) of L for all $t \in (\alpha, \beta]$.*

Proof. The proof is similar to the proof of Theorems 3.10 and 3.13. $\square$

The following example shows that there exist $\alpha, \beta \in (0, 1]$ with $\alpha < \beta$ such that μ is a sublattice (ideal, filter) with thresholds α, β which is not an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy sublattice (ideal, filter) of L.

Example 3.18. Consider the lattice L which is given in Example 3.5. Let μ be a fuzzy subset of L defined by:

$$\mu(0) = 1, \qquad \mu(a) = 0.6, \qquad \mu(b) = 0.8, \qquad \mu(1) = 0.4.$$

Then it is easy to check that μ is a fuzzy ideal with thresholds $\alpha = 0.6$, $\beta = 0.7$. If we take $k = 0.4$, then $\mu(a) \sqcap \mu(b) \not\leq \mu(a \vee b) \sqcup 0.3$. Hence μ is not an $(\overline{\in}, \overline{\in} \vee \overline{q_{0.4}})$-fuzzy ideal of L.

Theorem 3.19. *Let μ be a fuzzy subset of L and $\alpha, \beta \in (0, 1]$ with $\alpha < \beta$. Then*

(1) *μ is a fuzzy sublattice (ideal, filter) of L if and only if μ is a fuzzy sublattice (ideal, filter) with thresholds $\alpha = 0$, and $\beta = 1$;*
(2) *μ is an $(\in, \in \vee q_k)$-fuzzy sublattice (ideal, filter) of L if and only if μ is a fuzzy sublattice (ideal, filter) with thresholds $\alpha = 0$ and $\beta = \frac{1-k}{2}$;*
(3) *μ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy sublattice (ideal, filter) of L if and only if μ is a fuzzy sublattice (ideal, filter) with thresholds $\alpha = \frac{1-k}{2}$ and $\beta = 1$.*

Proof. Straightforward. $\square$

4. Implication-based fuzzy sublattice (ideal, filter)

Fuzzy logic is an extension of set theoretic multivalued logic in which the truth values are linguistic variables or terms of the linguistic variable truth. Some operators, for example, $\wedge, \vee, \neg, \rightarrow$ in fuzzy logic are also defined by using truth tables and the extension principle can be applied to derive definitions of the operators. In fuzzy logic, truth value of fuzzy proposition P is denoted by $[P]$. For a universe U of discourse, we display, the fuzzy logical and corresponding set-theoretical notations used in this paper [28].

$[x \in F] = F(x)$;
$[x \notin F] = 1 - F(x)$;
$[P \wedge Q] = \min\{[P], [Q]\}$;
$[P \vee Q] = \max\{[P], [Q]\}$;
$[P \rightarrow Q] = \min\{1, 1 - [P] + [Q]\}$;
$[\forall x P(x)] = \inf[P(x)]$;
$\models P$ if and only if $[P] = 1$ for all valuations.

The truth valuation rules given above are those in the £ukasiewicz system of continuous-valued logic. Of course, various implication operators have been defined. We only show a selection of them in the next table. α denotes the degree of truth (or degree of membership) of the premise, β the respective values for the consequence, and I the resulting degree of truth for the implication.

Name	Definition of implication operators
Early Zadeh	$I_m(\alpha, \beta) = \max\{1-\alpha, \min\{\alpha, \beta\}\}$
Łukasiewicz	$I_a(\alpha, \beta) = \min\{1, 1-\alpha+\beta\}$
Standard star (Gödel)	$I_g(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha \leq \beta \\ \beta & \text{if } \alpha > \beta \end{cases}$
Contraposition of Gödel	$I_{cg}(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha \leq \beta \\ 1-\alpha & \text{if } \alpha > \beta \end{cases}$
Gaines-Rescher	$I_{gr}(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha \leq \beta \\ 0 & \text{if } \alpha > \beta \end{cases}$
Kleene-Dienes	$I_b(\alpha, \beta) = \max\{1-\alpha, \beta\}$
Goguen	$I_{gg}(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha \leq \beta \\ \dfrac{\beta}{\alpha} & \text{if } \alpha > \beta \end{cases}$

The "quality" of these implication operators could be evaluated either empirically or axiomatically.

In the following, we considered the definition of implication operator in the Łukasiewicz system of continuous-valued logic.

Definition 4.1. A fuzzy subset μ of a lattice L satisfies:

(i) for any $x, y \in L, \models [\,[x \in \mu] \wedge [y \in \mu] \longrightarrow [x \vee y \in \mu\,]\,]$,
(ii) for any $x, y \in L, \models [\,[x \in \mu] \wedge [y \in \mu] \longrightarrow [x \wedge y \in \mu\,]\,]$,

then μ is called a *fuzzifying sublattice of* L.

μ is called a *fuzzifying ideal* of L if μ is a fuzzifying sublattice of L and

(iii) for any $x, y \in L, \models [\,[x \in \mu] \wedge [y \leq x] \longrightarrow [y \in \mu\,]\,]$.

μ is called a *fuzzifying filter* of L if μ is a fuzzifying sublattice of L and

(iv) for any $x, y \in L, \models [\,[y \in \mu] \wedge [y \leq x] \longrightarrow [x \in \mu\,]\,]$.

Now, we introduce the concept of t-tautology, i.e., $\models_t P$ if and only if $[P] \geq t$ for all valuations.

So, we can extend the concept of implication-based fuzzy sublattice (ideal, filter) in the following way.

Definition 4.2. Let μ be a fuzzy subset of a lattice L and $t \in (0, 1]$ is a fixed number. If

(i) for any $x, y \in L, \models_t [\,[x \in \mu] \wedge [y \in \mu] \longrightarrow [x \vee y \in \mu]\,]$,
(ii) for any $x, y \in L, \models_t [\,[x \in \mu] \wedge [y \in \mu] \longrightarrow [x \wedge y \in \mu]\,]$,

then μ is called a *t-implication-based fuzzy sublattice* of L. Moreover, if μ satisfies the condition:

(iii) for any $x, y \in L, \models_t [\,[x \in \mu] \wedge [y \leq x] \longrightarrow [y \in \mu]\,]$,

then μ is called a *t-implication-based fuzzy ideal* of L.

μ is called a *t-implication-based fuzzy filter* of L if μ is a t-implication-based fuzzy sublattice of L and

(iv) for any $x, y \in L, \models_t [\,[y \in \mu] \wedge [y \leq x] \longrightarrow [x \in \mu]\,]$.

Now, let I be an implication operator. Then we have the following corollary.

Corollary 4.3. *μ is a t-implication-based fuzzy sublattice of a lattice L if and only if*

(i) $I(\mu(x) \sqcap \mu(y), \mu(x \vee y)) \geq t$ *for all* $x, y \in L$,
(ii) $I(\mu(x) \sqcap \mu(y), \mu(x \wedge y)) \geq t$ *for all* $x, y \in L$.

μ is a t-implication-based fuzzy ideal of a lattice L if and only if μ is a t-implication-based fuzzy sublattice of L and

(iii) $I(\mu(x), \mu(y)) \geq t$ *with* $y \leq x$ *for all* $x, y \in L$.

μ is a t-implication-based fuzzy filter of a lattice L if and only if μ is a t-implication-based fuzzy sublattice of L and

(iv) $I(\mu(y), \mu(x)) \geq t$ *with* $y \leq x$ *for all* $x, y \in L$.

Theorem 4.4. *For any fuzzy subset μ of L, we have the following.*

(1) *If $I = I_{gr}$, then μ is a 0.5-implication-based fuzzy sublattice (ideal, filter) of L if and only if μ is a fuzzy sublattice (ideal, filter) of L.*

(2) *If* $I = I_g$, *then* μ *is a* $\frac{1-k}{2}$*-implication-based fuzzy sublattice (ideal, filter) of* L *if and only if* μ *is an* $(\in, \in \vee q_k)$*-fuzzy sublattice (ideal, filter) of* L.
(3) *If* $I = I_{cg}$, *then* μ *is a* $\frac{1-k}{2}$*-implication-based fuzzy sublattice (ideal, filter) of* L *if and only if* μ *is an* $(\overline{\in}, \overline{\in} \vee \overline{q_k})$*-fuzzy sublattice (ideal, filter) of* L *of* L.

Proof. We only prove (2) and the proofs (1) and (3) are similar.

(2) Assume that μ is a $\frac{1-k}{2}$-implication-based fuzzy sublattice of L. Then $I_g(\mu(x) \sqcap \mu(y), \mu(x \vee y)) \geq \frac{1-k}{2}$ and $I_g(\mu(x) \sqcap \mu(y), \mu(x \wedge y)) \geq \frac{1-k}{2}$. It follows that $\mu(x \vee y) \geq \mu(x) \sqcap \mu(y)$ or $\mu(x) \sqcap \mu(y) \geq \mu(x \vee y) \geq \frac{1-k}{2}$ and $\mu(x \wedge y) \geq \mu(x) \sqcap \mu(y)$ or $\mu(x) \sqcap \mu(y) \geq \mu(x \wedge y) \geq \frac{1-k}{2}$. Hence

$\mu(x \vee y) \sqcup 0 = \mu(x \vee y) \geq \mu(x) \sqcap \mu(y) \geq \frac{1-k}{2}$ and $\mu(x \wedge y) \sqcup 0 = \mu(x \wedge y) \geq \mu(x) \sqcap \mu(y) \geq \frac{1-k}{2}$.

Therefore μ is a fuzzy sublattice of L with thresholds $\alpha = 0$ and $\beta = \frac{1-k}{2}$ and hence μ is an $(\in, \in \vee q_k)$-fuzzy sublattice of L by Theorem 3.19.

The proof for the case that μ is a $\frac{1-k}{2}$-implication-based fuzzy ideal (filter) of L is similar.

Conversely, assume that μ is an $(\in, \in \vee q_k)$-fuzzy sublattice of L. Then

$$\mu(x \vee y) = \mu(x \vee y) \sqcup 0 \geq \mu(x) \sqcap \mu(y) \sqcap \frac{1-k}{2} \quad \text{and}$$
$$\mu(x \vee y) = \mu(x \vee y) \sqcup 0 \geq \mu(x) \sqcap \mu(y) \sqcap \frac{1-k}{2}.$$

For the first case, if $\mu(x) \sqcap \mu(y) \sqcap \frac{1-k}{2} = \mu(x) \sqcap \mu(y)$ then $\mu(x \vee y) \geq \mu(x) \sqcap \mu(y)$ and thus $I_g(\mu(x) \sqcap \mu(y), \mu(x \vee y)) = 1 \geq \frac{1-k}{2}$. Suppose that $\mu(x) \sqcap \mu(y) \sqcap \frac{1-k}{2} = \frac{1-k}{2}$. Then $\mu(x \vee y) \geq \frac{1-k}{2}$ and hence $I_g(\mu(x) \sqcap \mu(y), \mu(x \vee y)) \geq \mu(x \vee y) \geq \frac{1-k}{2}$. For the second case, if $\mu(x) \sqcap \mu(y) \sqcap \frac{1-k}{2} = \mu(x) \sqcap \mu(y)$ then $\mu(x \wedge y) \geq \mu(x) \sqcap \mu(y)$ and thus $I_g(\mu(x) \sqcap \mu(y), \mu(x \wedge y)) \geq = 1 \geq \frac{1-k}{2}$. Suppose that $\mu(x) \sqcap \mu(y) \sqcap \frac{1-k}{2} = \frac{1-k}{2}$. Then $\mu(x \wedge y) \geq \frac{1-k}{2}$ and hence $I_g(\mu(x) \sqcap \mu(y), \mu(x \wedge y)) \geq \mu(x \wedge y) \geq \frac{1-k}{2}$. Therefore μ is a $\frac{1-k}{2}$-implication-based fuzzy sublattice of L.

The proof for the case that μ is a $\frac{1-k}{2}$-implication-based fuzzy ideal (filter) of L is similar. □

5. Conclusion

To obtain a general type of an $(\in, \in \vee q)$-fuzzy sublattice (ideal, filter) of a lattice, we have introduced the notion of an $(\in, \in \vee q_k)$-fuzzy sublattice (ideal, filter). We have provided examples which are $(\in, \in \vee q_k)$-fuzzy sublattice (ideal, filter) but not an $(\in, \in \vee q)$-fuzzy sublattice (ideal, filter). We have dealt with characterizations of an $(\in, \in \vee q_k)$-fuzzy sublattice (ideal, filter) and an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy sublattice (ideal, filter). We have investigated conditions for an $(\in, \in \vee q_k)$-fuzzy sublattice (ideal, filter) (resp. $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy sublattice (ideal, filter)) to be a fuzzy sublattice (ideal, filter). Different classes of lattices are characterized by the properties of these fuzzy sublattices (ideals, filters). Using the notion of a fuzzy sublattice (ideal, filter) with thresholds, we have discussed characterizations of a fuzzy sublattice (ideal, filter), an $(\in, \in \vee q_k)$-fuzzy sublattice (ideal, filter) and an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy sublattice (ideal, filter). In this manner, we give some characterizations of three particular cases of lattices by these generalized fuzzy sublattices (ideals, filters). We finally have considered characterizations of a fuzzy sublattice (ideal, filter), $(\in, \in \vee q_k)$-fuzzy sublattice (ideal, filter) and an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy sublattice (ideal, filter) by using implication operators and the notion of implication-based fuzzy sublattice (ideal, filter).

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